MATH2020A Homework 3

(15.4)

3. Region:

$$\frac{\pi}{4} \leq \theta \leq \frac{3\pi}{4}, 0 \leq r \leq \frac{1}{\sin\theta}$$

5. Region:

$$0 \leq \theta \leq \frac{\pi}{6}, 1 \leq r \leq \frac{2\sqrt{3}}{\cos\theta}$$

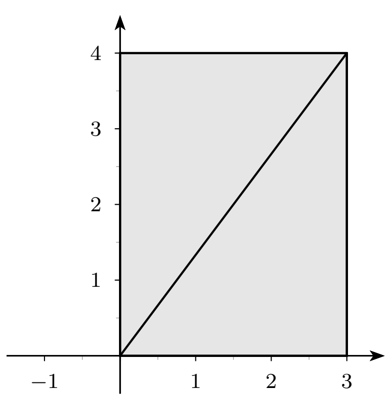
and

$$\frac{\pi}{6} \leq \theta \leq \frac{\pi}{2}, 1 \leq r \leq \frac{2}{\sin\theta}$$

18.

$$\begin{aligned} \int_{-1}^{1}\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \frac{2}{(1+x^2+y^2)^2} dydx &= \int_0^{2\pi}\int_0^1 \frac{2r}{(1+r^2)^2} drd\theta \\ &= \int_0^{2\pi} \left[-\frac{1}{1+r^2}\right]_0^1 d\theta = \pi \end{aligned}$$

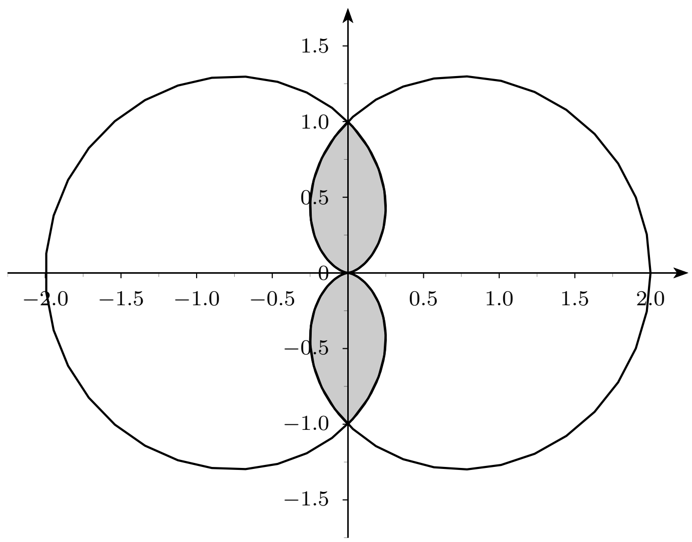
26. Sketch this region,



Then

$$\begin{aligned} & \int_0^{\tan^{-1}\frac{4}{3}} \int_0^{3\sec\theta} r^7 dr d\theta + \int_{\tan^{-1}\frac{4}{3}}^{\frac{\pi}{2}} \int_0^{4\csc\theta} r^7 dr d\theta \\ = & \int_0^3 \int_0^4 (x^2+y^2)^3 dy dx \end{aligned}$$

32. Sketch this region.



Then

$$\begin{aligned} A &= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \int_0^{1-\cos\theta} r dr d\theta + \int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} \int_0^{1+\cos\theta} r dr d\theta \\ &= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{(1-\cos\theta)^2}{2} d\theta + \int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} \frac{(1+\cos\theta)^2}{2} d\theta \\ &= -2 + \frac{3\pi}{4} - 2 + \frac{3\pi}{4} \\ &= -4 + \frac{3\pi}{2} \end{aligned}$$

44.

$$\begin{aligned} A &= \int_\alpha^\beta \int_0^{f(\theta)} r dr d\theta = \int_\alpha^\beta \left[\frac{r^2}{2}\right]_0^{f(\theta)} d\theta \\ &= \int_\alpha^\beta \frac{1}{2}[f(\theta)]^2 d\theta = \int_\alpha^\beta \frac{1}{2} r^2 d\theta \end{aligned}$$

(15.5)

19.

$$\begin{aligned}\int_0^{\frac{\pi}{4}}\int_0^{\ln\sec v}\int_{-\infty}^{2t} e^x\,dx\,dt\,dv &= \int_0^{\frac{\pi}{4}}\int_0^{\ln\sec v} e^{2t}\,dt\,dv\\ &= \int_0^{\frac{\pi}{4}}\left(\frac{1}{2}\sec^2 v-\frac{1}{2}\right)dv\\ &= \frac{1}{2}-\frac{\pi}{8}\end{aligned}$$

28.

$$\begin{aligned}V &= \int_0^1\int_0^{1-x}\int_0^{\cos(\frac{\pi x}{2})} dz\,dy\,dx = \int_0^1\int_0^{1-x}\cos(\frac{\pi x}{2})dy\,dx\\ &= \int_0^1(1-x)\cos(\frac{\pi x}{2})dx\\ &= \left[\frac{2}{\pi}(1-x)\sin(\frac{\pi x}{2})\right]_0^1+\frac{2}{\pi}\int_0^1\sin(\frac{\pi x}{2})dx\\ &= \left[\frac{4}{\pi^2}\sin(\frac{\pi x}{2})\right]_0^1=\frac{4}{\pi^2}\end{aligned}$$

31. Find shadow on yz-plane, $R=\{(y,z)|y\geq 0, z\geq 0, y^2+4z^2\leq 16\}$. So we have

$$V=\iint\limits_R\int_0^{4-y}dx\,dy\,dz=\iint\limits_R(4-y)dy\,dz$$

Change variable z by $z=\tilde{z}/2$, hence $dz=\frac{1}{2}\tilde{z}$. Notice the inequality $y^2+4z^2\leq 16$ will imply $y^2+\tilde{z}^2\leq 16$. So the region will become

$$\tilde{R}=\{(y,\tilde{z})|y\geq 0, z\geq 0, y^2+\tilde{z}^2\leq 16\}$$

Hence integration becomes

$$V=\iint\limits_{\tilde{R}}(4-y)\frac{1}{2}dy\,d\tilde{z}$$

Change it to polar coordinate, i.e., replace $dyd\tilde{z}$ by $rdrd\theta$, and replace $\sqrt{y^2+\tilde{z}^2}$ by r, region becomes

$$\overline{R} = \{(r,\theta)|0 \leq r \leq 4, 0 \leq \theta \leq \frac{\pi}{2}\}$$

As we notice, $y = r\cos\theta$ after this replacement. Hence

$$V = \frac{1}{2}\int_0^{\frac{\pi}{2}}\int_0^4 (4 - r\cos\theta)rdrd\theta = \int_0^{\frac{\pi}{2}}(16 - \frac{32}{3}\cos\theta)d\theta = 8\pi - \frac{32}{3}$$

Remark: There are some tricks here by this method. But you can still use traditional method to compute.

42.

$$\begin{aligned}
\int_0^1\int_0^1\int_{x^2}^1 12xze^{zy^2}dydxdz &= \int_0^1\int_0^1\int_0^{\sqrt{y}} 12xze^{zy^2}dxdydz \\
&= \int_0^1\int_0^1 6yze^{zy^2}dydz \\
&= \int_0^1 \left[3e^{zy^2}\right]_0^1 dz \\
&= \int_0^1 (3e^z - 3)dz \\
&= 3e - 6
\end{aligned}$$